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# A note on the extended Blasius equation

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**Abstract**

In a recent paper, the Blasius equation was extended to a nonlinear equation like $af''' + ff'' = 0$ with the prime denoting differentiation with respect to the similarity variable $\eta$ and $a$ being a constant parameter. The current note will show that the solution of the extended Blasius equation can be obtained from the original Blasius equation solution with a variable transformation technique. The observed phenomena in numerical solutions of previous published work are theoretically analyzed. The equation is also discussed for an arbitrary real parameter or complex parameters. It is further shown that the extended Blasius equation is a special form of the similarity equation of momentum boundary layers over a flat plate with a temperature dependent property.


*Keywords:* Blasius equation; Boundary layer flow; Flat plate; Numerical solution; Similarity equation

## 1. Introduction

Blasius was the first solving the boundary layer flow over a fixed flat plate by using similarity transformation techniques [1]. Following his pioneering work, many studies were made to investigate this interesting equation either from mathematical points of view [2–4] or from engineering

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perspectives [5–10]. In a recent paper [11], the Blasius equation was extended to

$$af'''(\eta) + f(\eta)f''(\eta) = 0, \tag{1}$$

with boundary conditions

$$f(0) = f'(0) = 0, \qquad \text{and} \qquad f'(\infty) = 1 \tag{2}$$

where a prime denotes differentiation with respect to $\eta$, and $a$ is a constant. This equation has a general form and includes the two forms of classical Blasius equation in the literature with $a = 1$ and $a = 2$. In that paper, a shooting method was used to solve the nonlinear equation for $1 \le a \le 2$. In this short note, we will show that the solution for an arbitrary value of $a$ can be obtained from the classical Blasius equation with $a = 1$ after a transformation. The equation can also be generalized to other parameter domains and even for $a$ being a function of $\eta$.

## 2. Function transformation

Consider the extended Blasius equation (1) with a free parameter $a$. It is assumed that $a$ is a positive number at the moment without loss of generality. Define a new transform function $f(\eta) = \beta F(\zeta)$ with $\zeta = \eta/\beta$. Substituting this transformation into Eq. (1) yields

$$\frac{a}{\beta^2}\dddot{F} + F\ddot{F} = 0 \tag{3}$$

with boundary conditions

$$F(0) = \dot{F}(0) = 0, \qquad \text{and} \qquad \dot{F}(\infty) = 1, \tag{4}$$

where a dot denotes differentiation with respect to $\zeta$. By setting $\beta = \sqrt{a}$, one obtains

$$\dddot{F} + F\ddot{F} = 0 \tag{5}$$

which is exactly the same as a special format of the Blasius equation [1]. Then we obtain the solution of Eq. (1) in terms of $F(\zeta)$ as

$$f(\eta, a) = \sqrt{a}F(\eta/\sqrt{a}), \qquad f'(\eta, a) = \dot{F}(\eta/\sqrt{a}), \qquad \text{and} \qquad f''(\eta, a) = \frac{1}{\sqrt{a}}\ddot{F}(\eta/\sqrt{a}). \tag{6}$$

It is also straightforward to obtain the initial value $f''(0) = \frac{\ddot{F}(0)}{\sqrt{a}}$ at $\eta = 0$. The solution of Eq. (5) with boundary conditions (4) is well known [1] with $\ddot{F}(0) = 0.46960$. The solution of Eq. (1) can be integrated as an initial value problem. The observed phenomena in Ref. [11] can be explained based on Eq. (6). It is seen that increasing $a$ will decrease $f(\eta)$ and $f'(\eta)$. It is also shown that $f''(\eta)$ at any point near zero decreases with increasing $a$. For the points away from $\eta = 0$, $f''(\eta)$ increases with increasing $a$. For Eq. (5), it is known that $F(\zeta)$ and $\dot{F}(\zeta)$ are monotonically increasing functions and $\ddot{F}(\zeta)$ is a monotonically decreasing function of $\zeta$.

From Eq. (6) it is obtained that

$$\frac{\partial f(\eta, a)}{\partial a} = \frac{1}{2\sqrt{a}}F(\eta/\sqrt{a}) - \frac{\eta}{2a}\dot{F}(\eta/\sqrt{a}) = \frac{1}{2\sqrt{a}}[F(\zeta) - \zeta\dot{F}(\zeta)], \tag{7a}$$

$$\frac{\partial f'(\eta, a)}{\partial a} = -\frac{\eta}{2a^{3/2}}\ddot{F}(\zeta), \tag{7b}$$

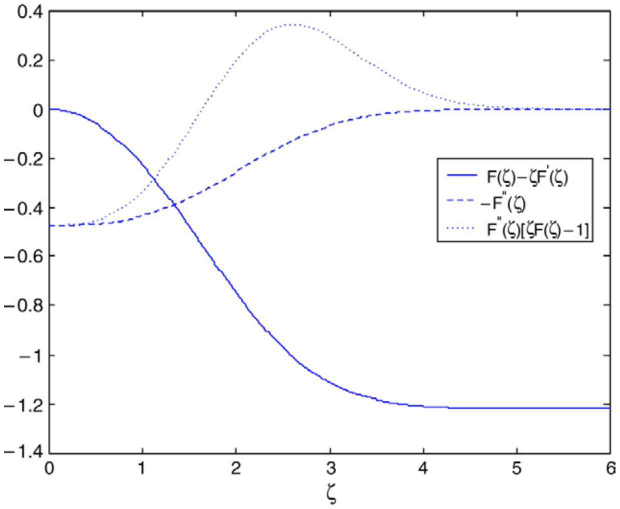


Fig. 1. The plots of $F(\zeta) - \zeta\dot{F}(\zeta)$, $-\ddot{F}(\zeta)$, and $\ddot{F}(\zeta)[\zeta F(\zeta) - 1]$.

$$\frac{\partial f''(\eta, a)}{\partial a} = -\frac{1}{2a^{3/2}}\ddot{F}(\eta/\sqrt{a}) - \frac{\eta}{2a^2}\dddot{F}(\eta/\sqrt{a}) = \frac{\ddot{F}(\zeta)}{2a^{3/2}}[\zeta F(\zeta) - 1]. \tag{7c}$$

The plots of $F(\zeta) - \zeta\dot{F}(\zeta)$, $-\ddot{F}(\zeta)$, and $\ddot{F}(\zeta)[\zeta F(\zeta) - 1]$ are shown in Fig. 1. It is seen that Eq. (7a) is always less than zero except at $\eta \neq 0$. Eq. (7b) is also less than zero for small $\eta$ and approaches zero for large $\eta$. Therefore, $f(\eta)$ and $f'(\eta)$ will decrease with increasing $a$ for $\eta \neq 0$. The value of Eq. (7c), however, will be less than zero first and then become greater than zero with increase of $\zeta$. Thus, we know that for the given $\eta$ near zero, $f''(\eta)$ will decrease with increasing $a$ and, for large enough $\eta$, $f''(\eta)$ will increase with increasing $a$. However, for very large $\eta$, the solution variation with $a$ becomes small because $\ddot{F}(\zeta)[\zeta F(\zeta) - 1]$ is approaching zero.

## 3. Further discussions

If $a$ is set to be zero, then Eq. (1) becomes

$$f(\eta) f''(\eta) = 0. \tag{8}$$

Because the order of Eq. (8) is two, a boundary condition will not be satisfied. Trivial solutions of Eq. (8) are $f(\eta) = 0$ or $f(\eta) = \eta$. As a matter of fact, when $a$ approaches zero, the solution of $f'(\eta)$ will approach a unit step function of $\eta$ as shown in Fig. 2 for different values of $a$. Another interesting observation is that when $a < 0$, there is no solution for Eq. (1) with the boundary conditions (Eq. (2)) by the following analysis. From Eq. (1) it is obtained that

$$f''(\eta) = f''(0)\mathrm{e}^{-\frac{1}{a}\int_0^\eta f(t)\,\mathrm{d}t}. \tag{9}$$

Because an exponential function is always positive, the variation behavior is determined by the sign of $f''(0)$. If $f''(0) > 0$, then $f'(\eta)$ and $f(\eta)$ will be monotonically increasing functions and will be positive in the domain $\eta \in (0, \infty)$. For a negative value of $a$ the exponential function part in Eq. (9) will go to infinity with increase of $\eta$; there is no finite value for $f'(\infty)$. On the other hand, when $f''(0) < 0$, $f'(\eta)$ and $f(\eta)$ will be monotonically decreasing functions and will be negative in the domain $\eta \in (0, \infty)$.

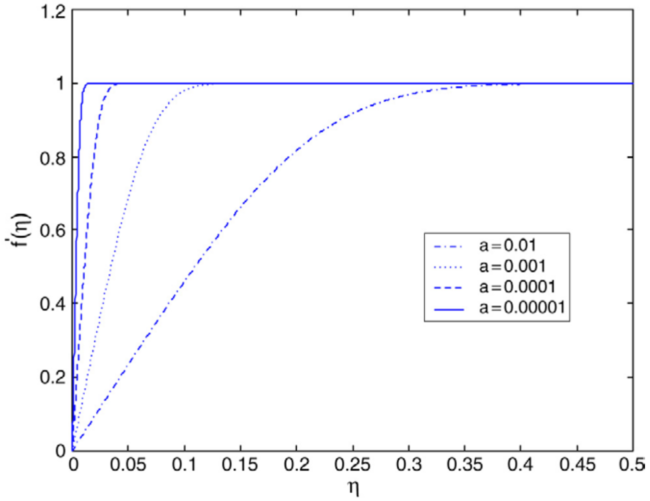


Fig. 2. The asymptotic behaviors of the extended Blasius equation solution for small values of $a$.

There will be a finite value of $f'(\infty)$ because $f''(\infty) \to 0$ for this case. However, the finite value of $f'(\infty)$ will be a negative number, which does not satisfy the boundary conditions of Eq. (2). Another case worthy of discussion is the case when $a \to \infty$. For this case, it is known that $\zeta \to 0$ for not very large $\eta$. Therefore, the series approximation solution of $F(\zeta)$ can be used for small $\zeta$. The series approximation solution of $F(\zeta)$ reads [12]

$$F(\zeta) = \frac{\sigma\zeta^2}{2!} - \frac{\sigma^2\zeta^5}{5!} + \frac{11\sigma^3\zeta^8}{8!} - \frac{375\sigma^4\zeta^{11}}{11!} + \cdots, \tag{10}$$

where $\sigma = \ddot{F}(0) = 0.46960$. Then the solution of Eq. (1) can be obtained by substituting Eq. (10) into (6) as follows:

$$f(\eta, a) = \frac{1}{a^{1/2}}\frac{\sigma\eta^2}{2!} - \frac{1}{a^2}\frac{\sigma^2\eta^5}{5!} + \frac{1}{a^{7/2}}\frac{11\sigma^3\eta^8}{8!} - \frac{1}{a^5}\frac{375\sigma^4\eta^{11}}{11!} + \cdots, \tag{11a}$$

$$f'(\eta, a) = \frac{\sigma\eta}{a^{1/2}} - \frac{1}{a^2}\frac{\sigma^2\eta^4}{4!} + \frac{1}{a^{7/2}}\frac{11\sigma^3\eta^7}{7!} - \frac{1}{a^5}\frac{375\sigma^4\eta^{10}}{10!} + \cdots, \tag{11b}$$

$$f''(\eta, a) = \frac{\sigma}{a^{1/2}} - \frac{1}{a^2}\frac{\sigma^2\eta^3}{3!} + \frac{1}{a^{7/2}}\frac{11\sigma^3\eta^6}{6!} - \frac{1}{a^5}\frac{375\sigma^4\eta^9}{9!} + \cdots. \tag{11c}$$

Mathematically speaking, when $a$ is a complex number, the solution of Eqs. (1) and (2) can be sought by separating the complex number into a real part and an imaginary part, as $a = a_R + \mathrm{i}a_I$ and $f(\eta) = f_R(\eta) + \mathrm{i}\, f_I(\eta)$ where $\mathrm{i} = \sqrt{-1}$. Then two equations will be obtained as follows:

$$a_R f_R''' + f_R f_R'' - (a_I f_I''' + f_I f_I'') = 0,\ \ f_R(0) = 0,\ \ f_R'(0) = 0,\ \text{and}\ f_R'(\infty) = 1 \tag{12a}$$

$$a_R f_I''' + f_R f_I'' + a_I f_R''' + f_I f_R'' = 0,\ \ f_I(0) = 0,\ \ f_I'(0) = 0,\ \text{and}\ f_I'(\infty) = 0. \tag{12b}$$

The above equations can be solved by shooting methods simultaneously.

Another generalization of the Blasius equation can be found in boundary layer flows with temperature dependent viscosity for engineering applications [13]. A general equation was shown as

$$[a(\eta) f''(\eta)]' + f(\eta) f''(\eta) = 0, \tag{13}$$

with boundary conditions

$$f(0) = f'(0) = 0, \qquad \text{and} \qquad f'(\infty) = 1. \tag{14}$$

It is seen that for such a situation the coefficient is not a constant any more; it is a function of $\eta$. When $a$ is a constant, this will reduce to the situation with temperature independent viscosity for the classical Blasius flat plate problem. More complex behaviors can be found for this generalized equation, and the equation can only be solved numerically.

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